Formal Measures of Dynamical Properties: Tipping Points and Robustness

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Motivation

- The idea of tipping points is important for topics as diverse as segregation, marketing, rioting, and global warming.
- Robustness considerations have extended beyond engineering and ecology to social institutions, political regimes, computer algorithms, decision procedures, and many more.
- These are not single, unambiguous concepts.
- The multiple distinct phenomena in these families of behavior are referred to using the various terms interchangedly.
- These distinct concepts have not been generally and formally defined, and doing so can make the terms' uses across these various applications consistent and comparable.

Motivation

- Complex systems analyses need to capture processes rather than a series of static snapshots.
- Existing statistical techniques are ill-suited to measure properties of system dynamics.
- Developing techniques to measure these features could provide a means to compare models across disciplines.
- Provides a conceptual benefit for understanding and categorizing system behaviors.
- Makes the inventor very popular at social gatherings.

Outline of Talk

- Comments on the general methodological approach.
- Building a Markov model from system output.
- Example using the Schelling Segregation agent-based model.
- Identifying the features of general system dynamics.
- Defining and measuring tipping point phenomena.
- Defining and measuring robustness family phenomena.
- Closing remarks and future work.

General Approach

- This technique is part of a larger project to develop new analysis methods for measuring system behaviors.
- Represent the system features, output data, interactions, etc. in a way that captures the systems' dynamics in a static structure.
- Apply existing measures from **that** static structure to this representation to reveal novel aspects of system behavior.
- Use insights gained from this translation to develop new measures for the adapted static structure.
- Feed these back to the original model for additional insights, and foster development of analogous measures in other representations.

Building a Markov Model

- Markov models are comprised of a set of states and the probabilistically weighted transitions among those states.
- The Markov model representation must be built in a specific way to run the properties of system dynamics analysis.
  - A state in the Markov model is a complete specification of the Q aspects of one configuration of the system.

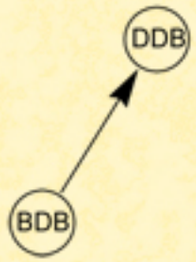
$$S_i = \{X_{1(i)}, X_{2(i)}, \ldots X_{Q(i)}\}$$

  - Example: If our system is an iterated game played by six players each with four possible actions then each state of the system has six aspects and each aspect takes on one of four values. That is $S_i = \{a(P1_{(i)}), a(P2_{(i)}), \ldots, a(P6_{(i)})\}$ and a particular state S_3 might be $\{a_3, a_2, a_3, a_1, a_4, a_3\}$.

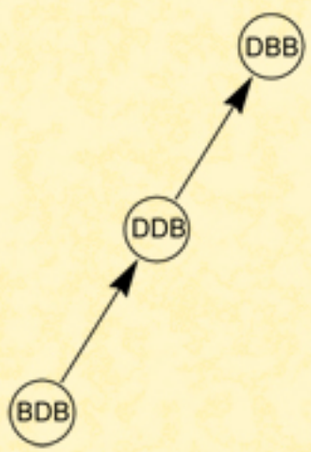
Building a Markov Model

BDB

Building a Markov Model



Building a Markov Model



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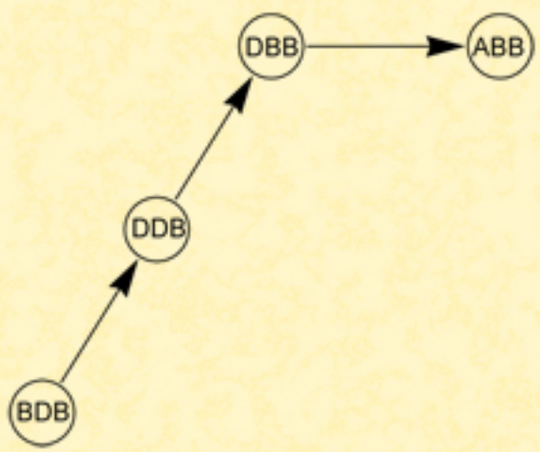

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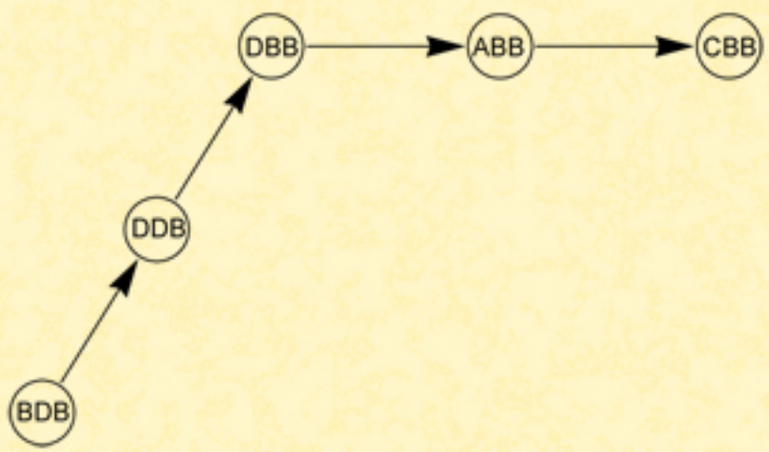

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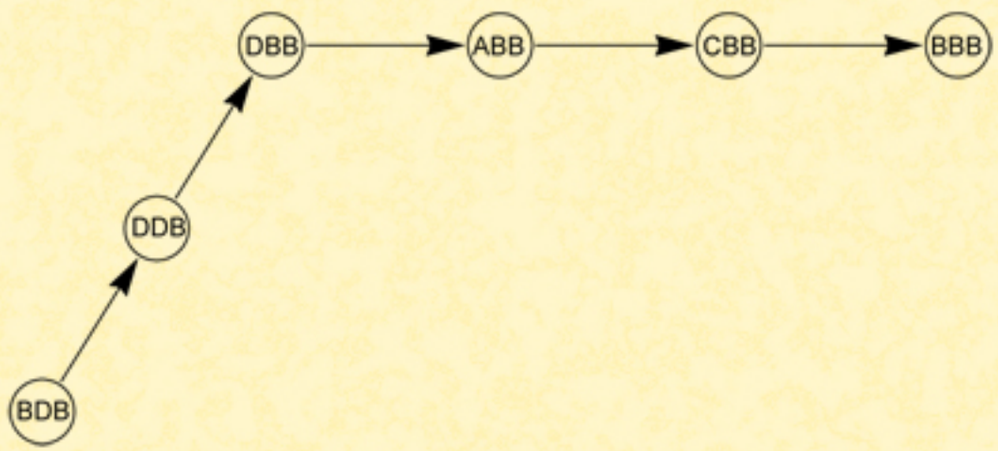

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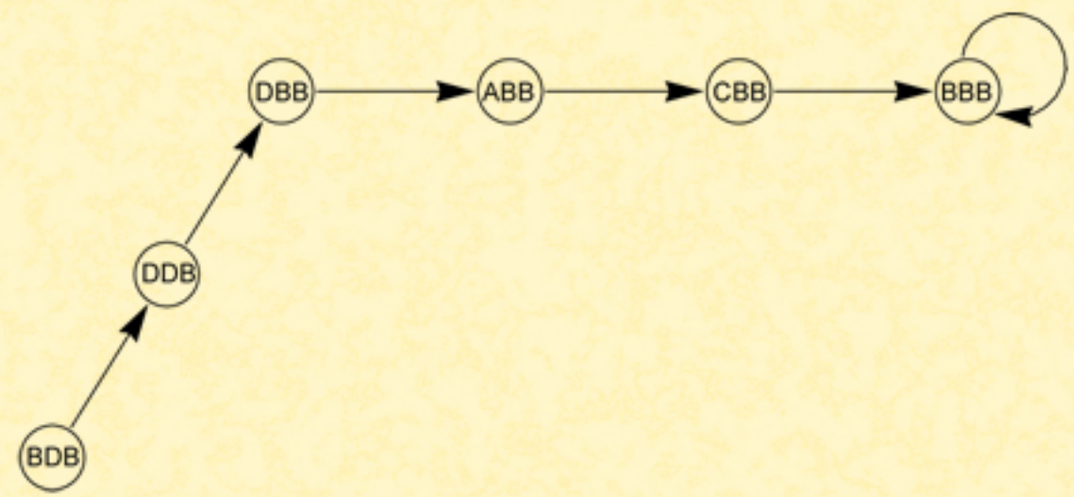

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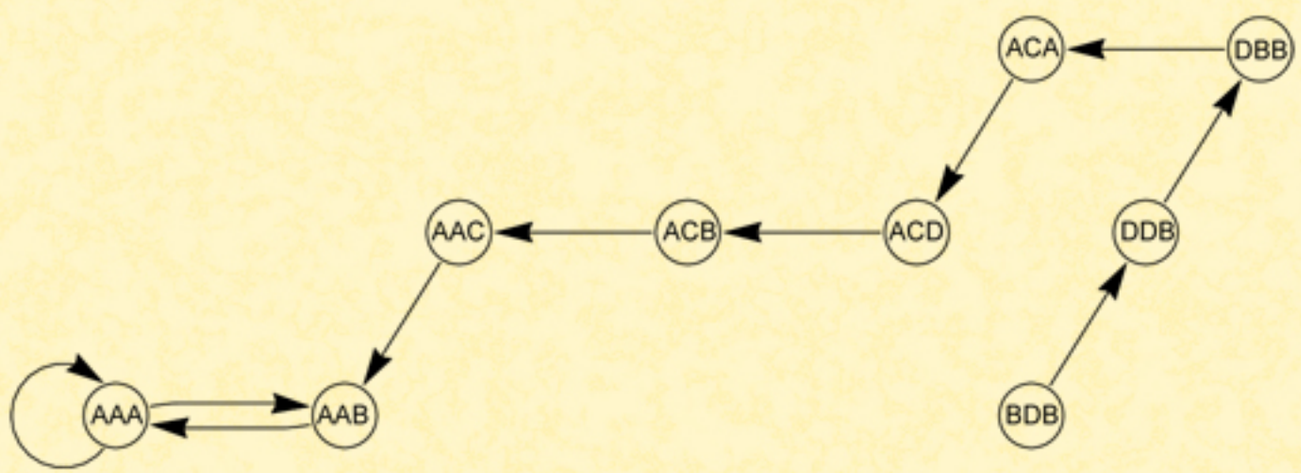

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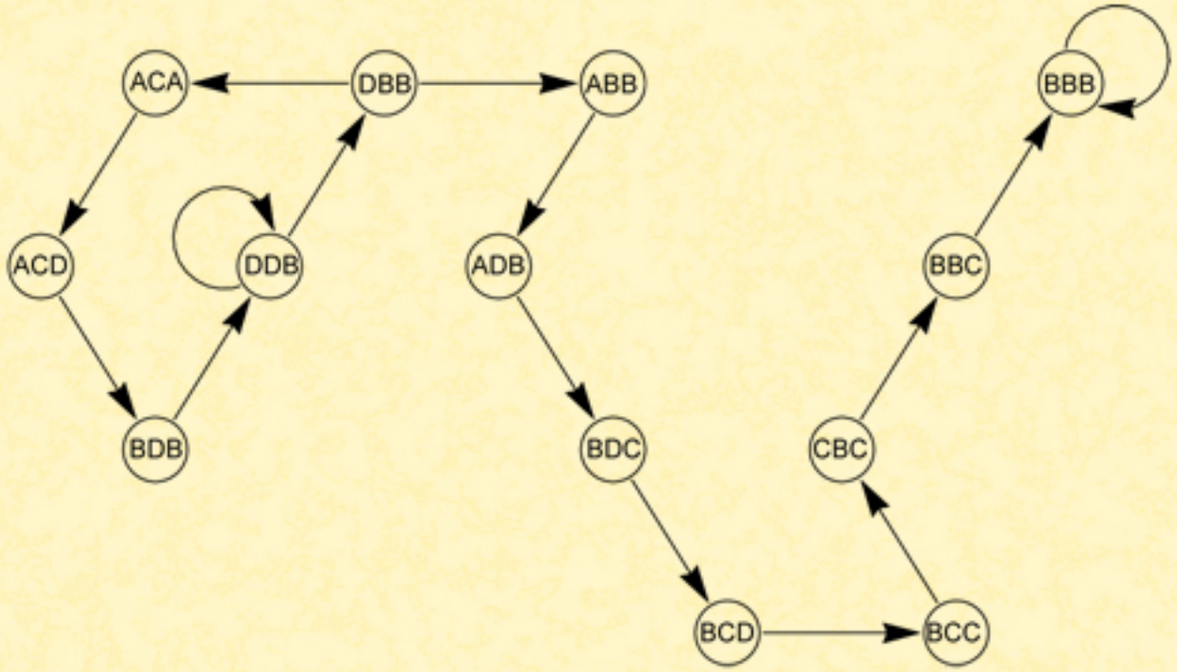

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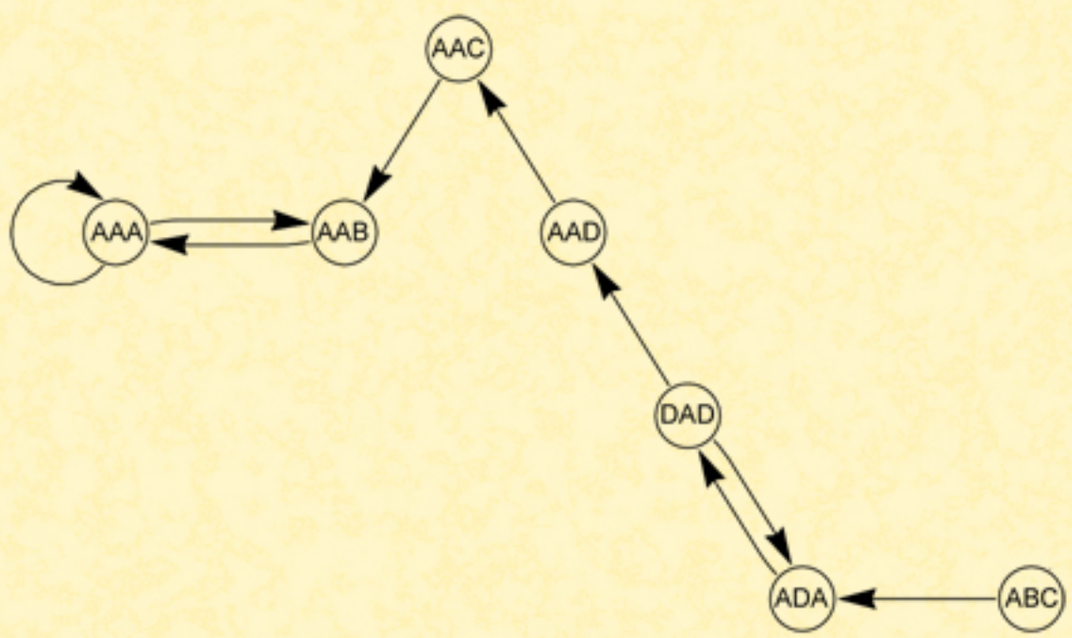
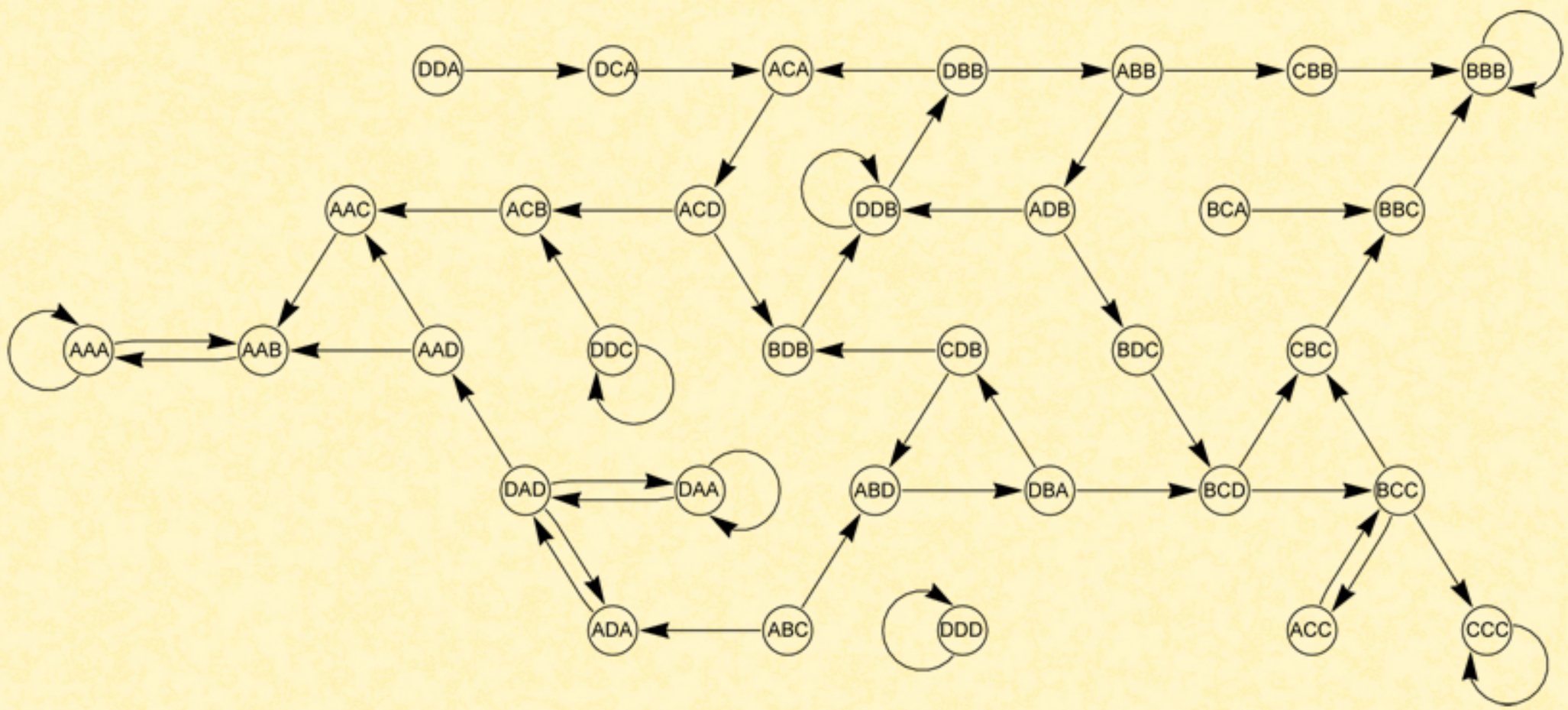
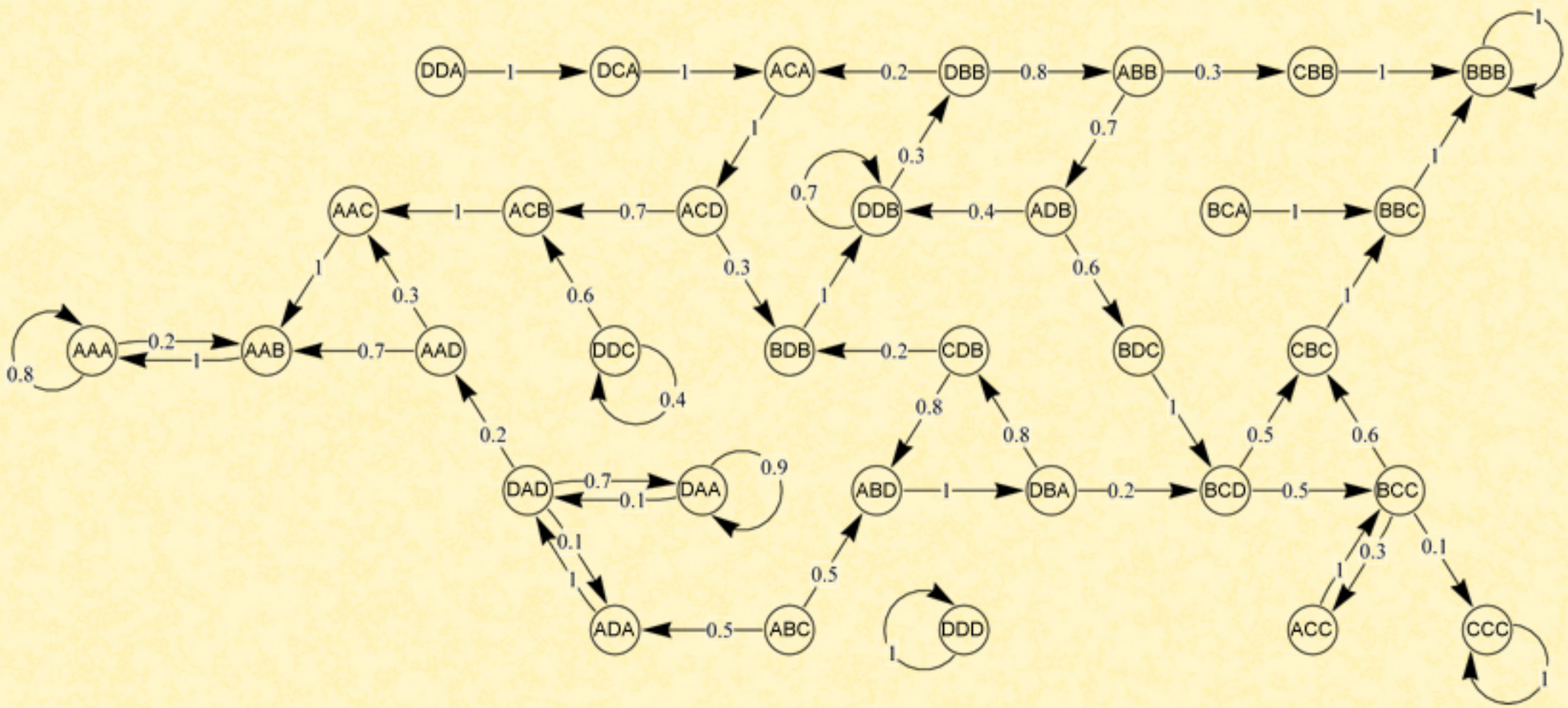


Building a Markov Model

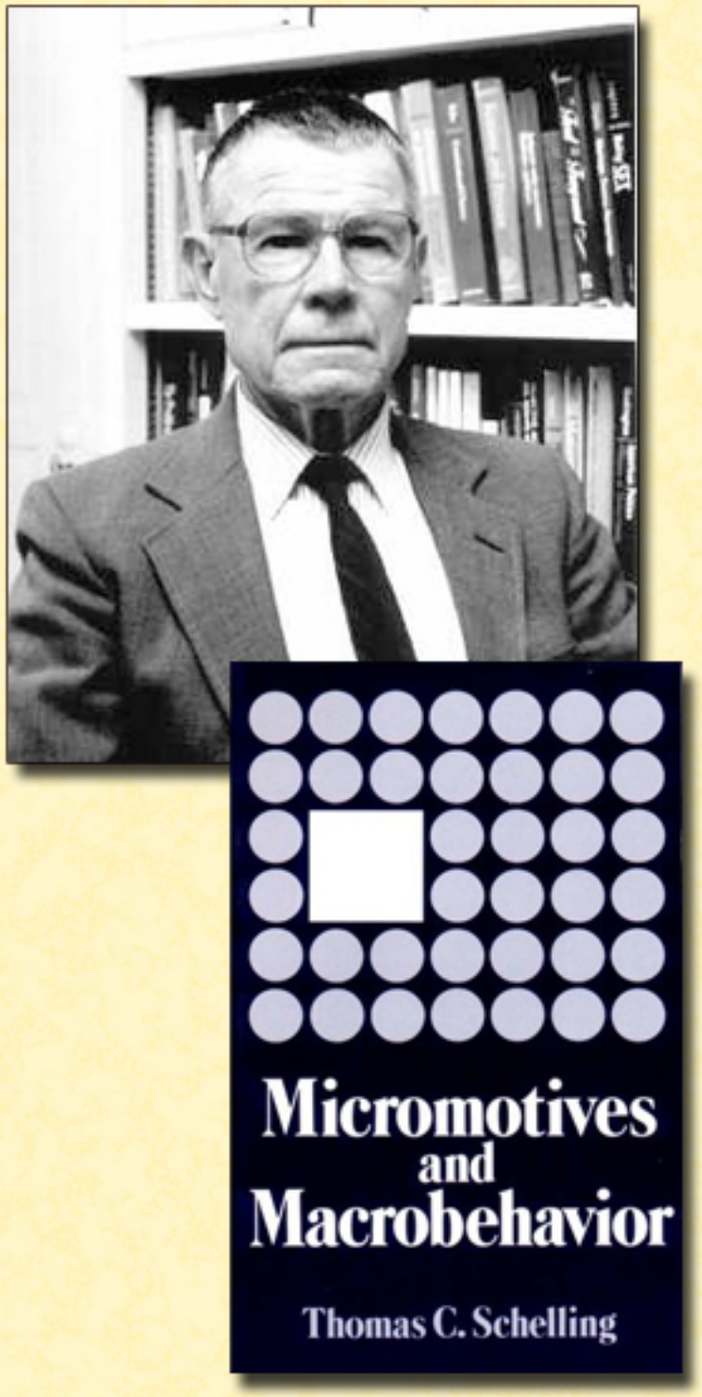


Building a Markov Model

- And that's not all!!! By using the frequency of transitions from your data you can produce the desired Markov model representation of your system's dynamics.



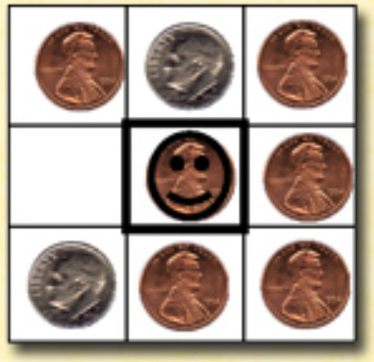
Thomas C. Schelling

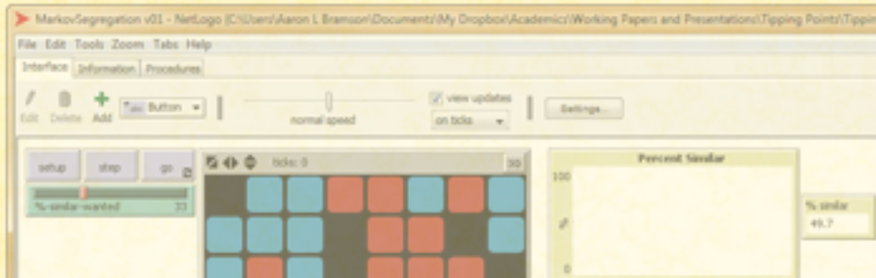


- Father of Agent-Based Modeling
- 2005 Nobel Prize Winner in Economics for work in Game Theory
- Clearly explained and motivated bottoms-ups approach in his 1978 book "Micromotives and Macrobehavior"
- Famous Segregation (aka "Tipping") model is the first modern agent-based model

Schelling Segregation Model

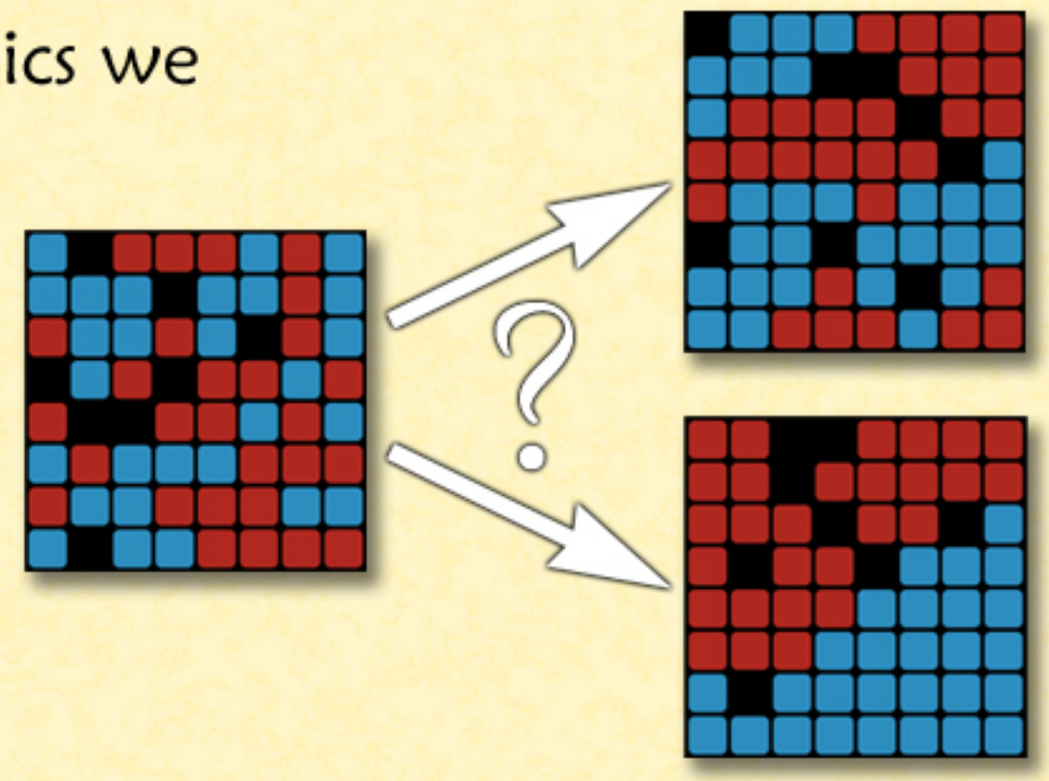
- There are two types of agents and they have a percentage of neighboring agents that they want to be of the same type.
- The neighbors are agents in any of the eight surrounding spaces of the grid.
- Unhappy agents move around to empty spaces until they are happy.
- Once all the agents are happy the model stops running.





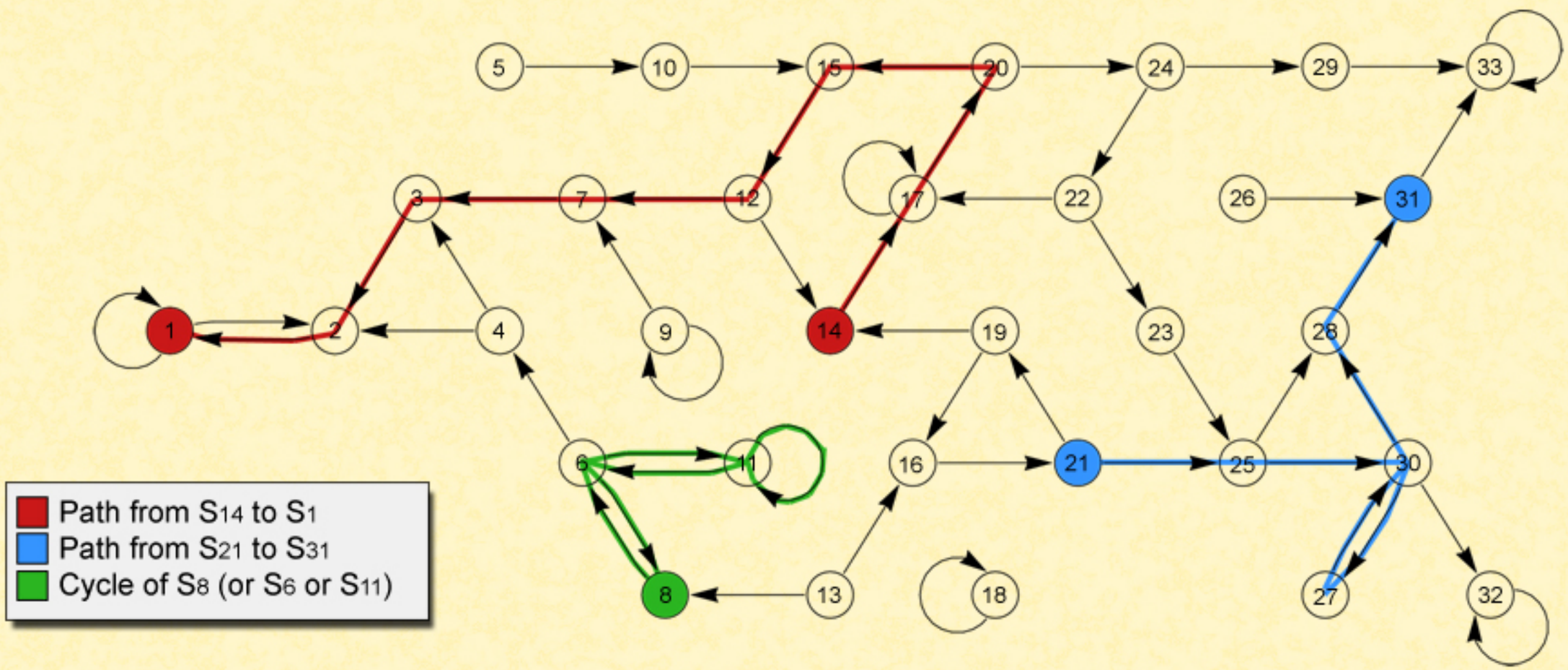
Schelling Segregation Model

- We want to foster greater diversity by managing to outcomes with less segregation.
- In order to influence the dynamics we must first understand them.
- Which initial configurations lead to which outcomes?
- Which states play pivotal roles in determining outcomes?
- Given the current actual state, what can we expect the outcome to be?



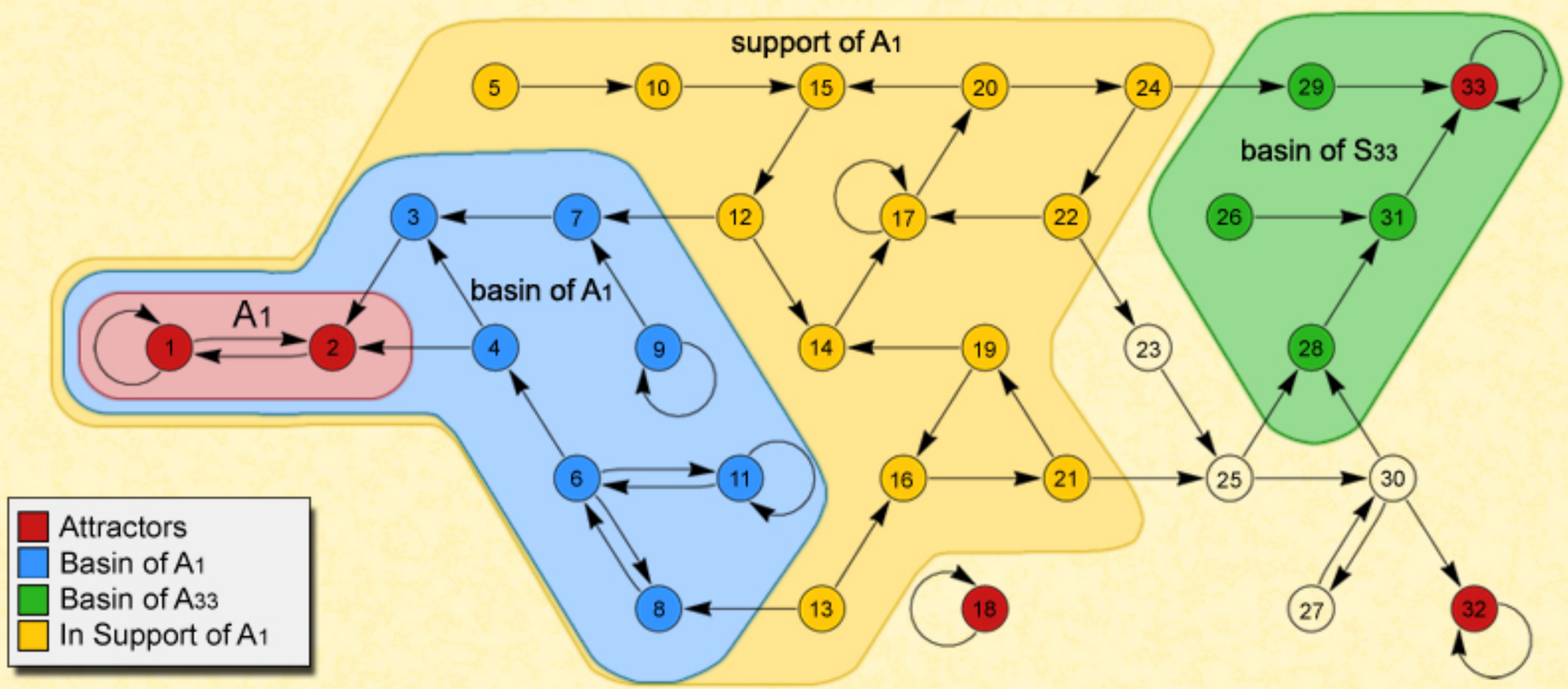
Paths and Cycles

- A path is an ordered collection of states and transitions such that from each state there exists a positive probability to transition to the successor state within the collection. A path from S_i to S_j denoted $\tilde{S}(S_i, S_j)$ is the set of states $\boldsymbol{S}$ such that
  (i) $s_0 = S_i$ in $\boldsymbol{S}$
  (ii) There exists T such that for all $t < T$ $P(s_{t+1} \text{ in } \boldsymbol{S} \mid s_t \text{ in } \boldsymbol{S}) > 0$
  (iii) $s_T = S_j$ in $\boldsymbol{S}$



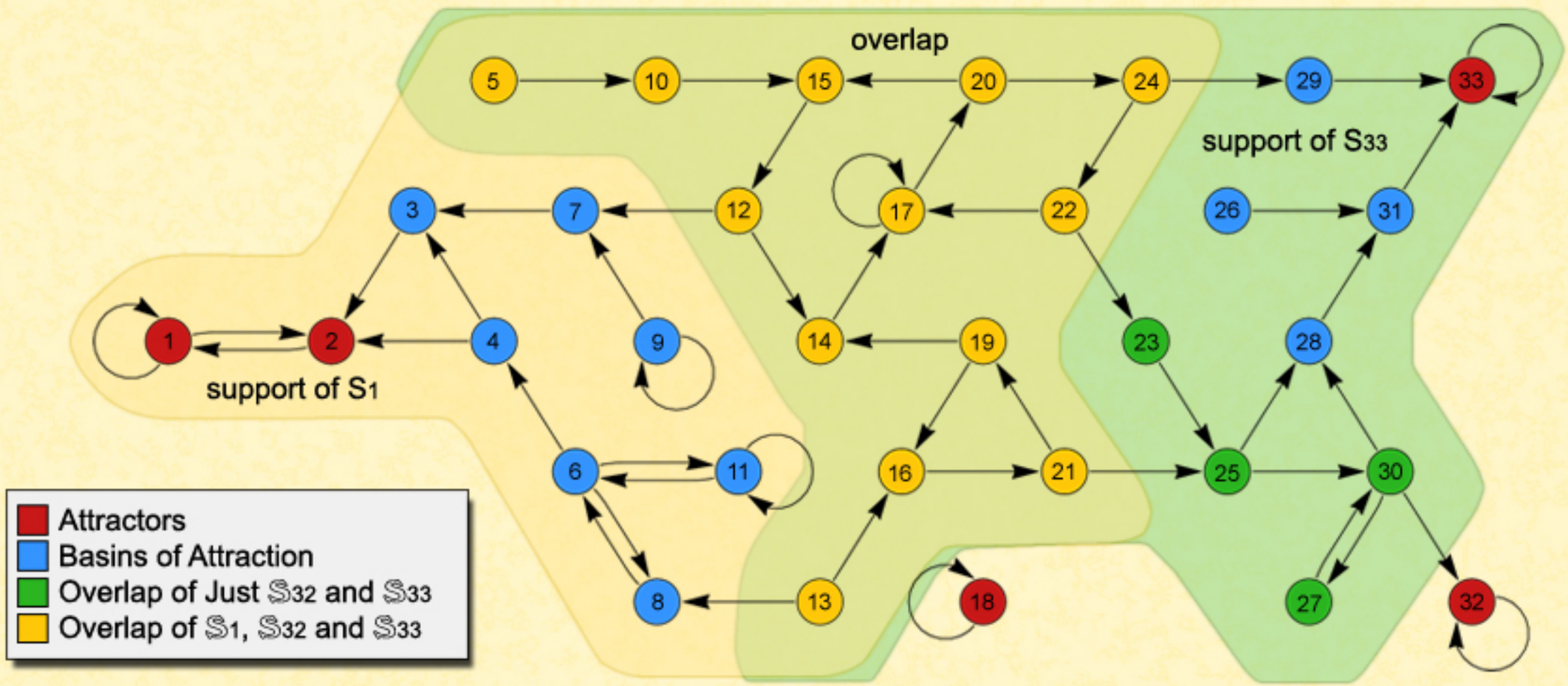
Landmarks in System Dynamics

- A system state that always transitions to itself is called an equilibrium or stable state. An equilibrium e_i is a state S_i such that $P(s_{t+1} = S_i | s_t = S_i) = 1$.
- Those states from which the system will eventually move into a specific attractor are said to be in that attractor's basin of attraction. The basin of A_i or $\mathbf{B}(A_i)$ is a set of states $\mathbf{S}$ such that there exists an $h \geq 0$ $P(s_{t+h} = A_i | s_t \text{ in } \mathbf{S}) = 1$.
- The support of a state (also known as its in-component) is the set of states which have a path to it.



Landmarks in System Dynamics

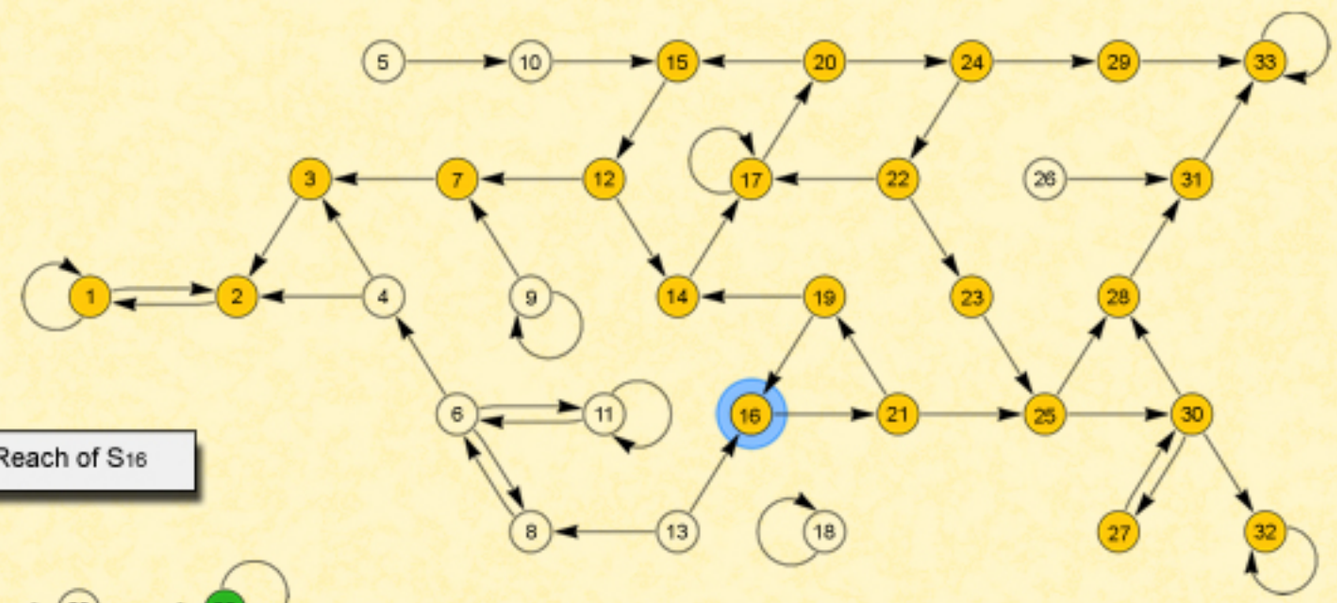
- The overlap of a collection of states is the set of states in all of their in-components (i.e. the intersection of thier supports).
- A state's out-degree is the number of distinct successor states (states that may be immediate transitioned into). The out-degree k_i of state S_i equals $|\{S_j : P(s_{t+1} = S_j \mid s_t = S_i) > 0\}|$.
- S_k will be used to denote a neighboring state and $\mathbf{S}_k$ the set of neighboring states.

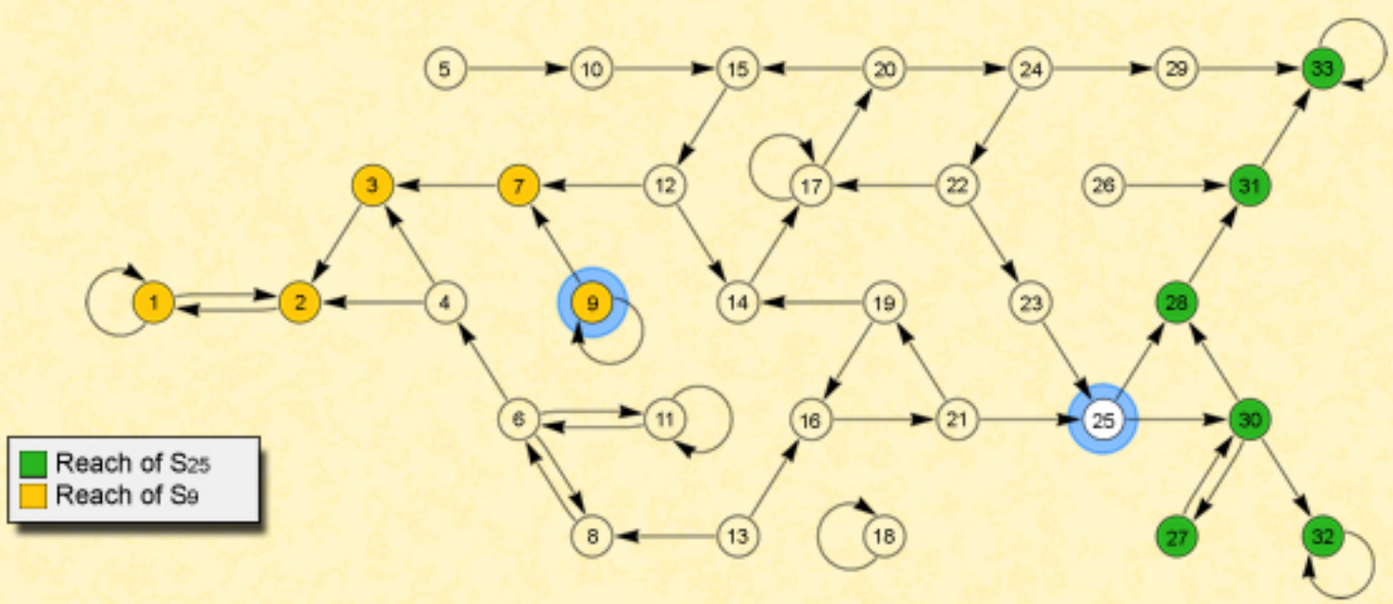


Landmarks in System Dynamics

- The reach of a state (also called its out-component and written $\mathbf{R}(S_i)$) is the set of states that the system may enter by following some sequence of transitions; i.e. all possible future states given an initial state.

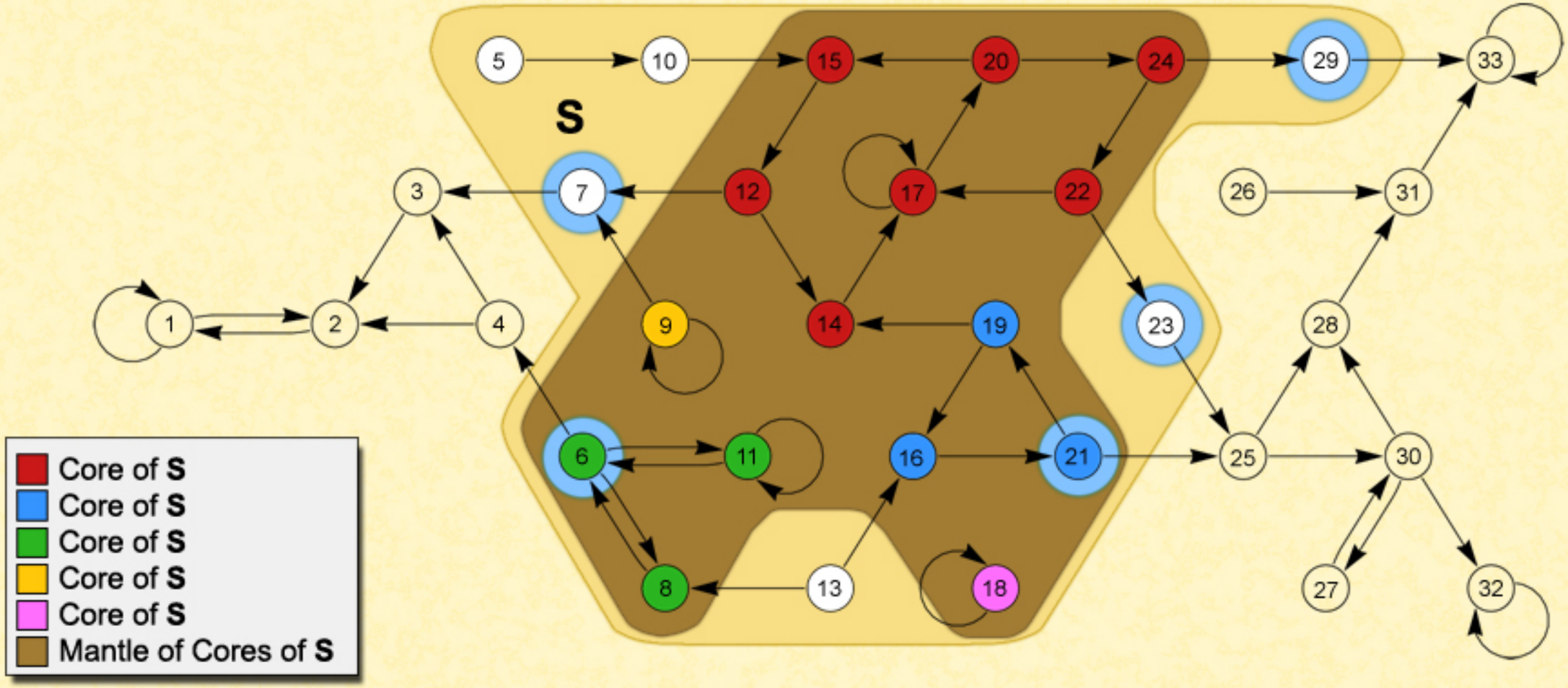
- Every successor state's reach is less than or equal to the previous state's reach. For every i and j, $\sim(S_i, S_j)$ implies $|\mathbf{R}(S_i)| \geq |\mathbf{R}(S_j)|$.





Landmarks in System Dynamics

- A core of a set is a subset wherein every member of the subset is in the reach of every member of the subset. This is the strongly connected component of a selected collection of states.
- Some sets will have multiple cores – the set of *S*'s cores can be called *S*'s mantle.
- The perimeter of a set is the collection of those states in the set that may transition to states outside the set. That is, ***S*** such that $P(s_{t+1} \text{ not in } \boldsymbol{S} \mid s_t \text{ in } \boldsymbol{S}) > 0$

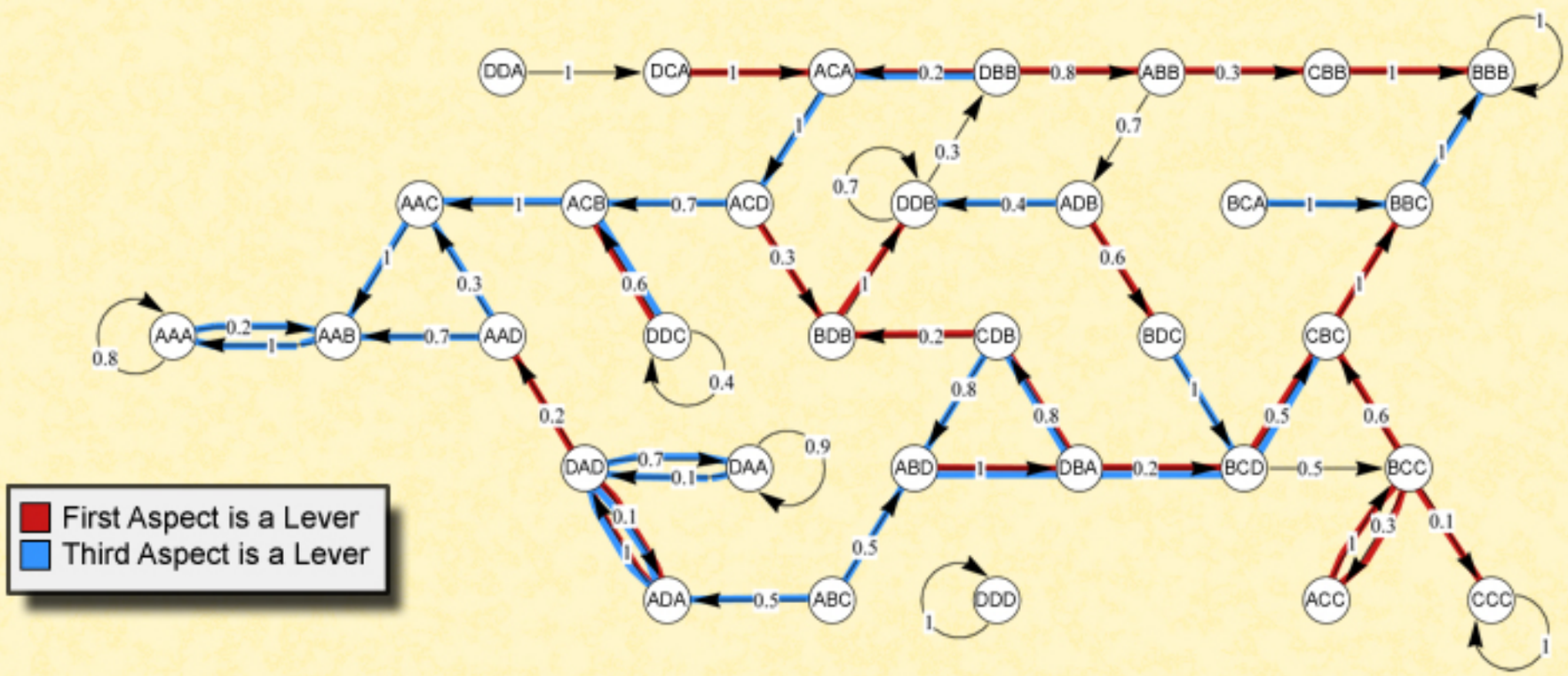


Summary of Landmarks

- Attractor - positive recurrent set of states
- Basin - set including exactly one attractor
- Support - set having a path to some focal state or set
- Reach - the states that some focal state has a path to
- Core - positive recurrent subset of a set
- Mantle - set of cores of a set
- Perimeter - states with transitions exiting a set

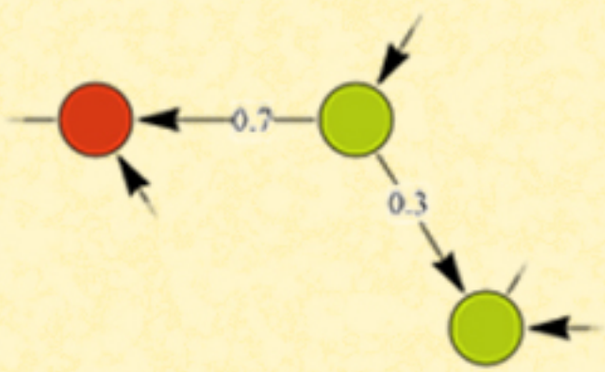
Levers and Thresholds

- The levers of a state are the aspects of a state such that a change in those aspects is suffcient to change the system's state.
- A lever point is a transition resulting from a change in a particular aspect (or set of aspects).
- The strength of a lever is the sum of the probabilities of all transitions that result from changing that lever.
- A threshold or threshold point is a particular value for a lever such that if the value of the property crosses the threshold value, it generates a transition.



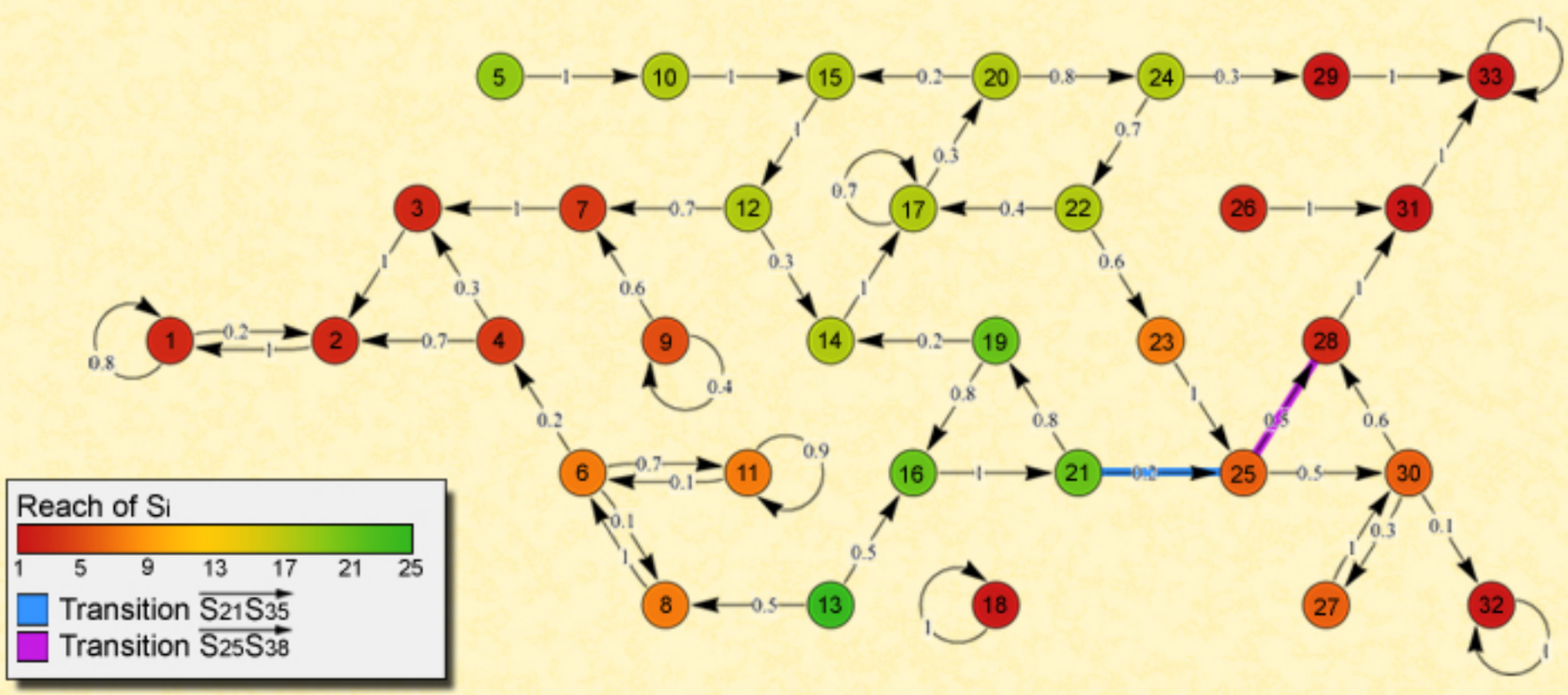
Critical Behavior

- A transition is considered critical behavior for a property if and only if it produces a decrease in the value of that property; $\sim(S_i, S_j)$ such that $V(S_i) > V(S_j)$.
- The critical-gap of a transition is the change in value across a transition.
- A **transition's** criticality is one minus the ratio of the start and end states' value. The criticality of $\sim(S_i, S_j)$ equals $1 - V(Sj) / V(Si)$.
- The criticality of a **state** is the probabilistically weighted sum of the criticality of all the transitions from that state.



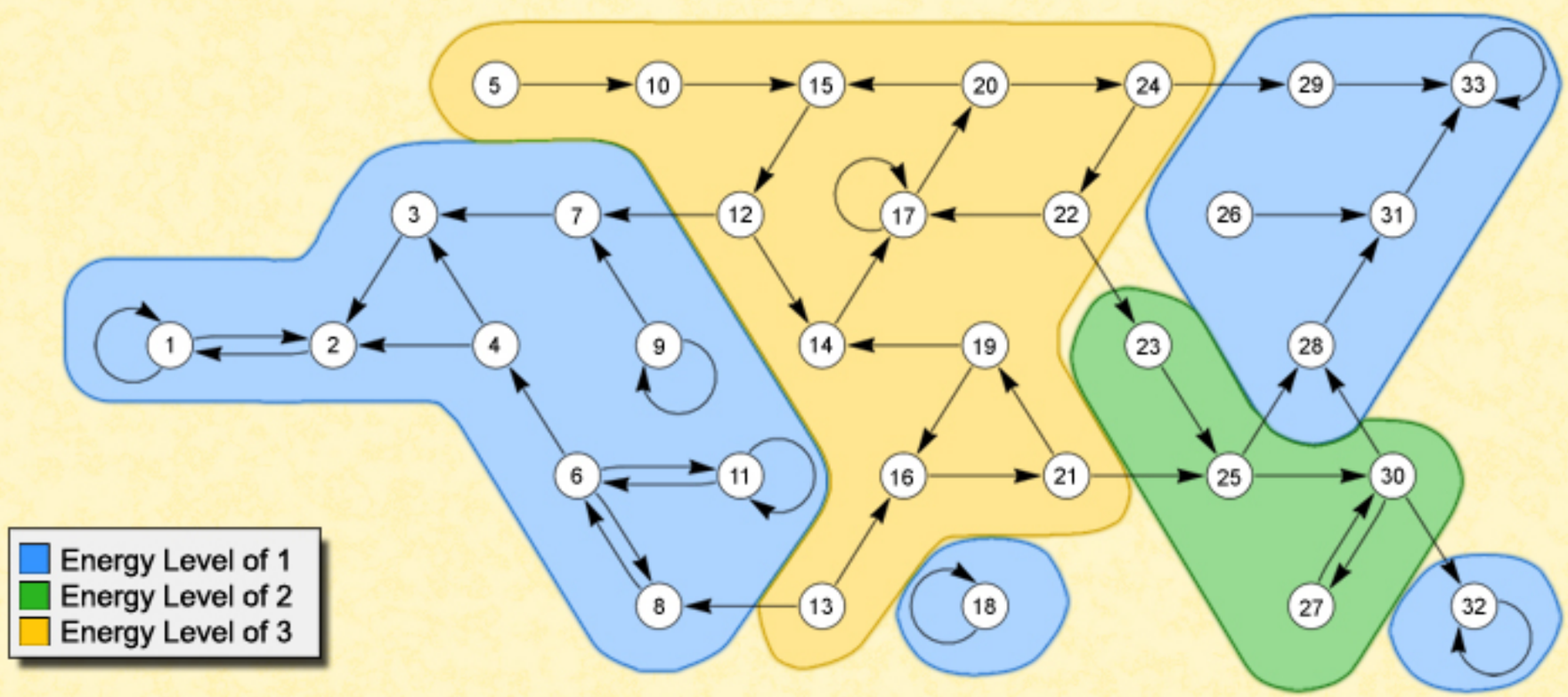
Critical Behavior

- As an example we can find the critical behavior of a decrease in reach; i.e. $\sim(S_i, S_j)$ such that $|R(S_i)| > |R(S_j)|$.
- The reach-gap of a transition is the change in the percent of the total number of states that can be reached. This quantity equals $|R(S_i)| / N - |R(S_j)| / N$.
- A **transition's** reach criticality of $\sim(S_i, S_j)$ equals $1 - |R(Sj)| / |R(Si)|$.
- The reach criticality of a **state** is the probabilistically weighted sum of the criticality of all the transitions from that state.



Tipping Behavior

- A tipping point is a state which is in the perimeter of an equivalence class for some property. Different properties reveal different kinds of tips.
- The energy level of a state is the number of reference states (e.g. attractors, functional states, states with high criticality) within its reach. Energy levels partition the system's states into equivalence classes.
- The tippiness of a state S_i is the probabilistically weighted proportional decreases in some property over its immediate successors: 1 − Sum over j, P_{ij} ($V(S_j)$ / $V(S_i)$).



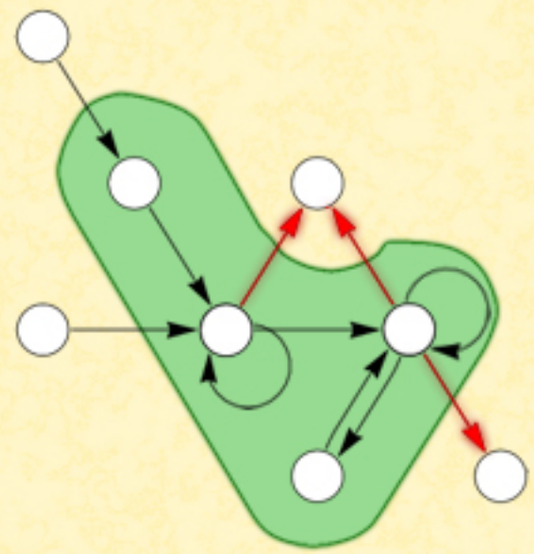
Summary of Tipping Behavior

- Lever- state aspects changes that mark state changes.
- Lever Strength- the probability mass of a lever's state changes.
- Threshold- a value for an aspect that generates a state change.
- Criticality- the change in aspects across states.
- Tipping Point- states that may lose a property through a transition.
- Energy Level- the number of reachable states with a property.
- Tippiness- the expected amount of property loss.

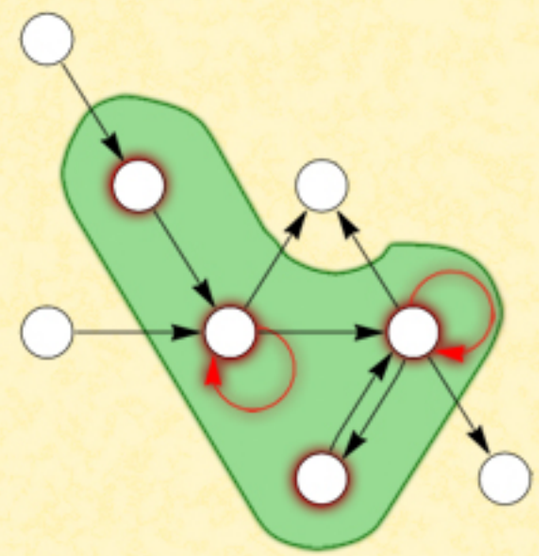
Stable and Static

- A state's stability is how likely that state is to self-transition: $P(s_{t+1} = S_i | s_t = S_i)$.
- The stability of a **set** is the probability that the system will not transition out of the set given that the system starts within the set. We calculate this as the average of the individual states' **exit probabilities**.
- The degree to which a set is static is the average of the states' stability values.

stability



set stability

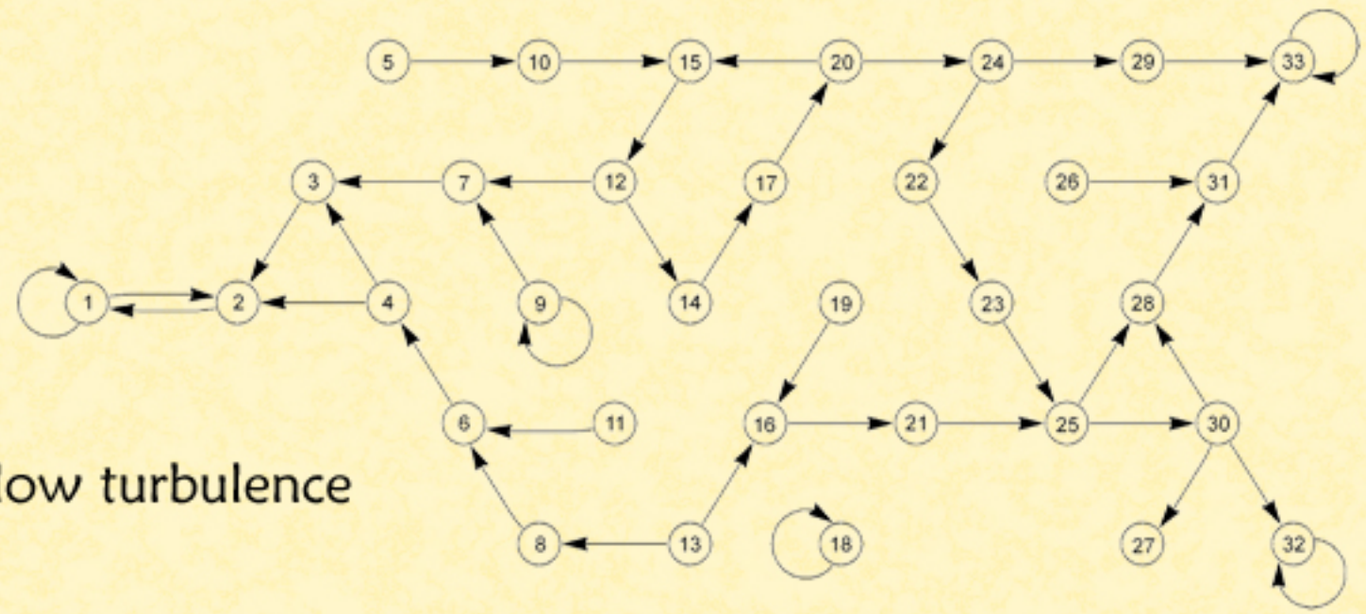


set staticness

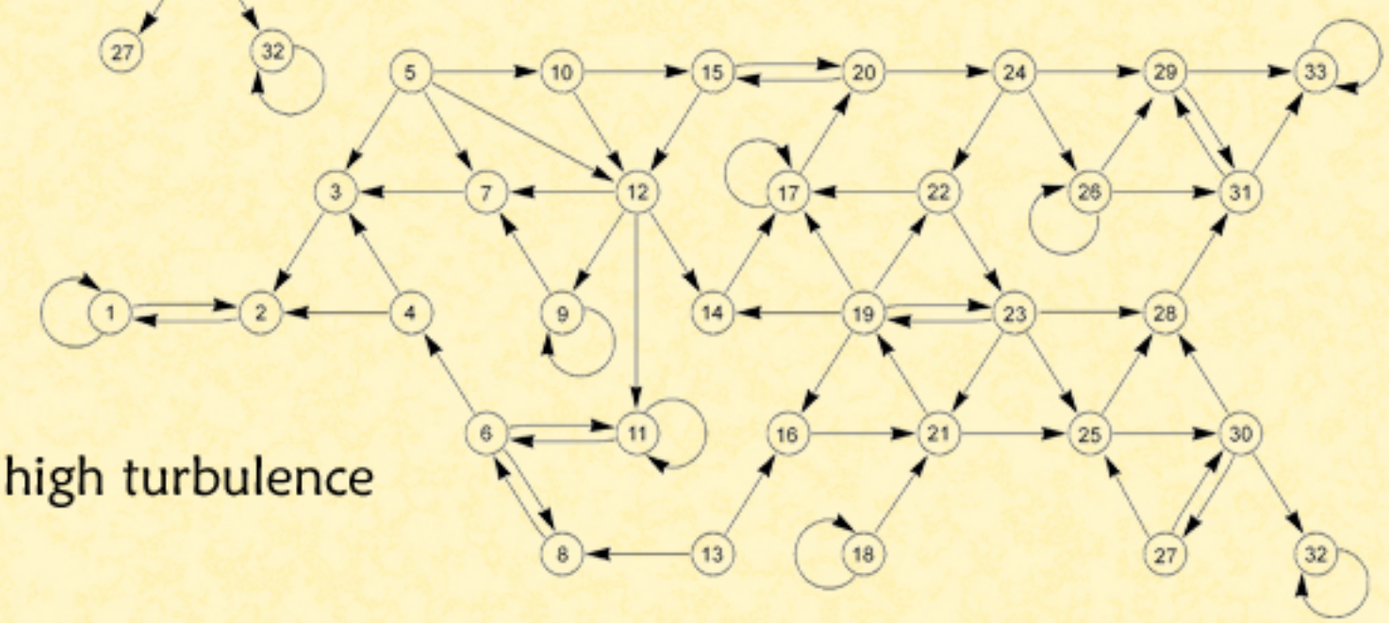
Turbulence

- The turbulence of a set is the average percentage of states that its states can transition into. We can calculate **S**'s turbulence with the average ratio of each state's degree to the number of states in **S**: $1/|\mathbf{S}|$ Sum over S_i (k / $|\mathbf{S}|$).

- As a refinement of turbulence, weighted turbulence of the state S_i equals zero if $k = 1$ and for $k > 1$ can be calculated as Sum over j of $1 - (P(\sim(S_i, S_j)) - 1/k)^2$.



low turbulence



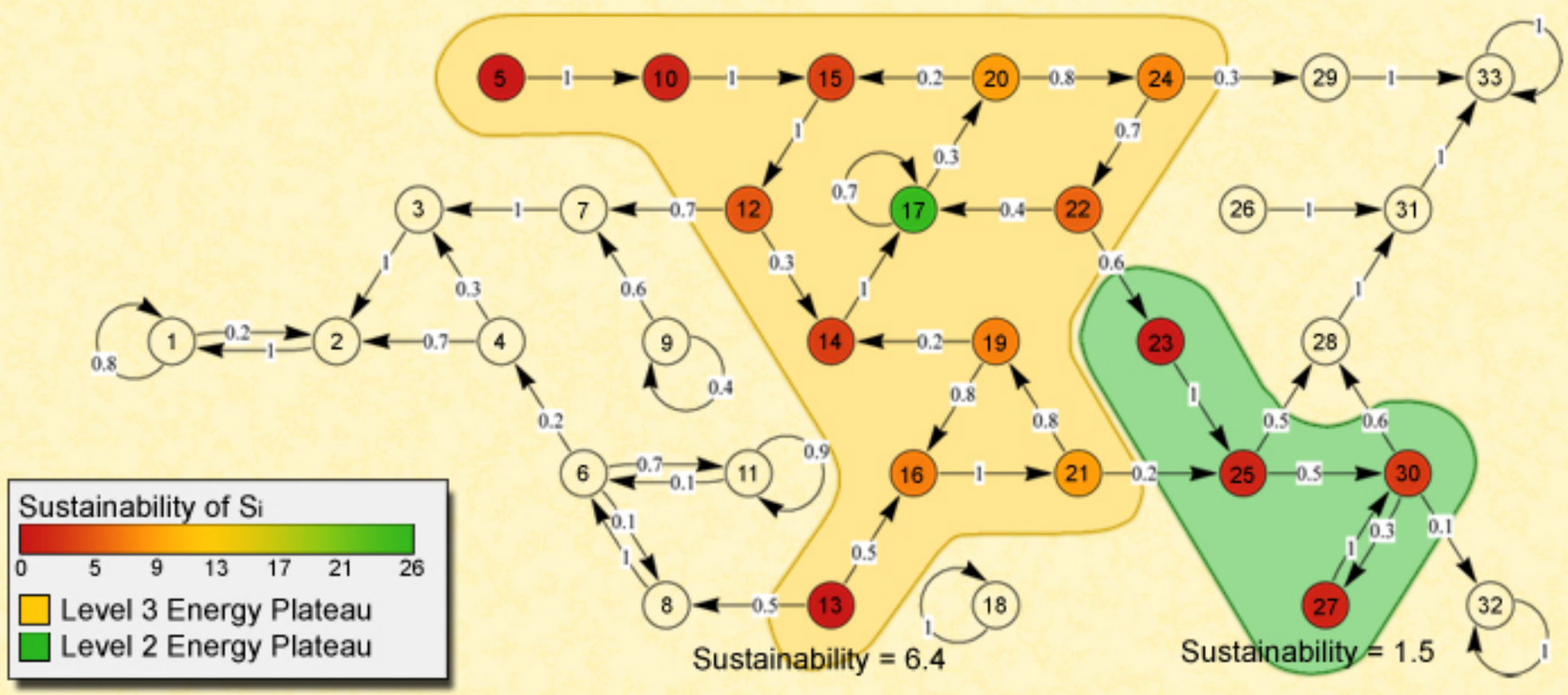
high turbulence

Sustainable and Susceptible

- The sustainability of ***S*** is the average cumulative long-term probability density of future states that remain in the set starting from each state in the set: $1/|S|$ Sum over S_i, Sum over t $P(s_{t+1}$ in $\boldsymbol{S} | s_t$ in $\boldsymbol{S})$.

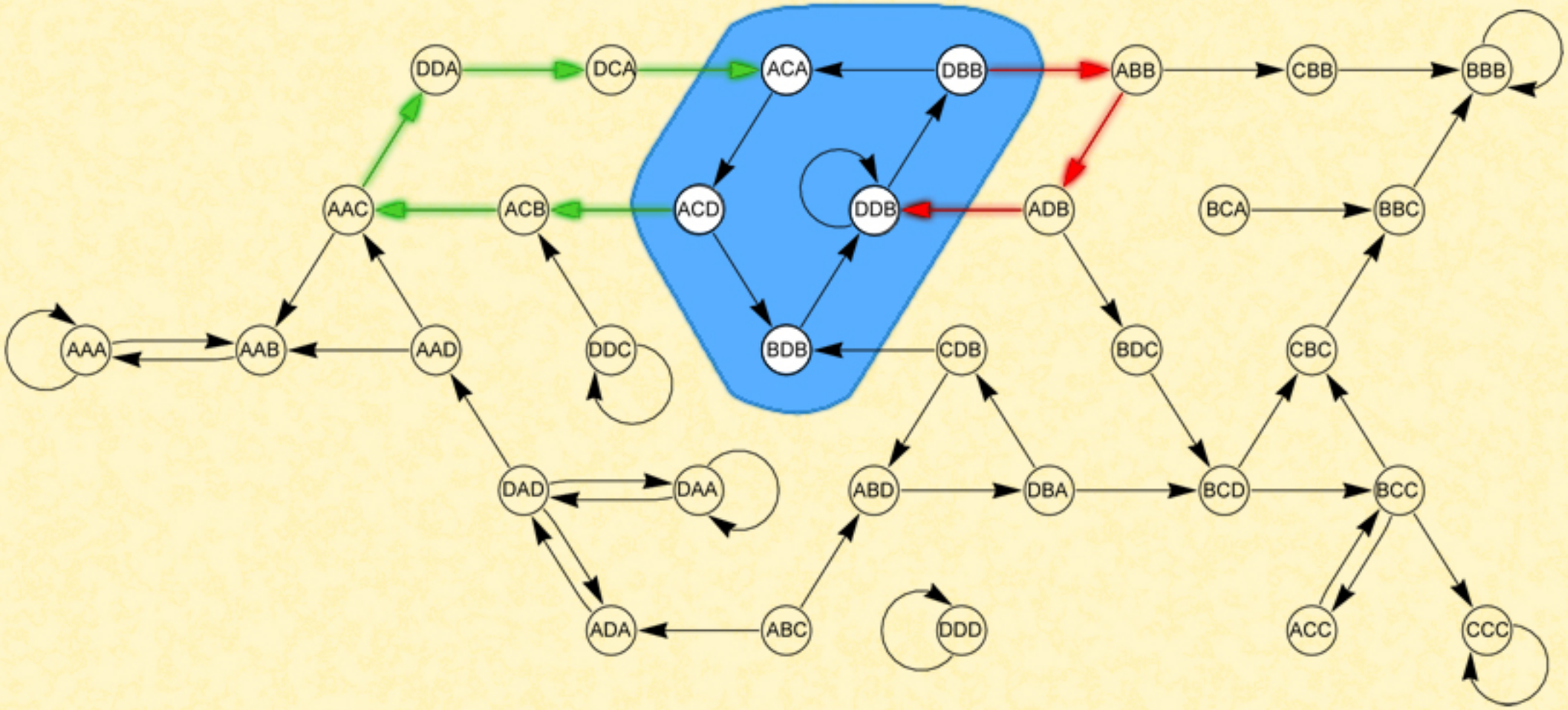
- The degree to which ***S*** is susceptible to S_i is how much more (or less) likely it is to transition out of ***S*** conditional on it being in a particular state S_i of ***S***: Sum over t $P(s_{t+1}$ in $\boldsymbol{S} | s_t$ in $\boldsymbol{S}$ and $s_0 = S_i)$ − sustainability of ***S.***

- Given this definition we can see that a positive susceptibility means a lower probability to stay within ***S***.



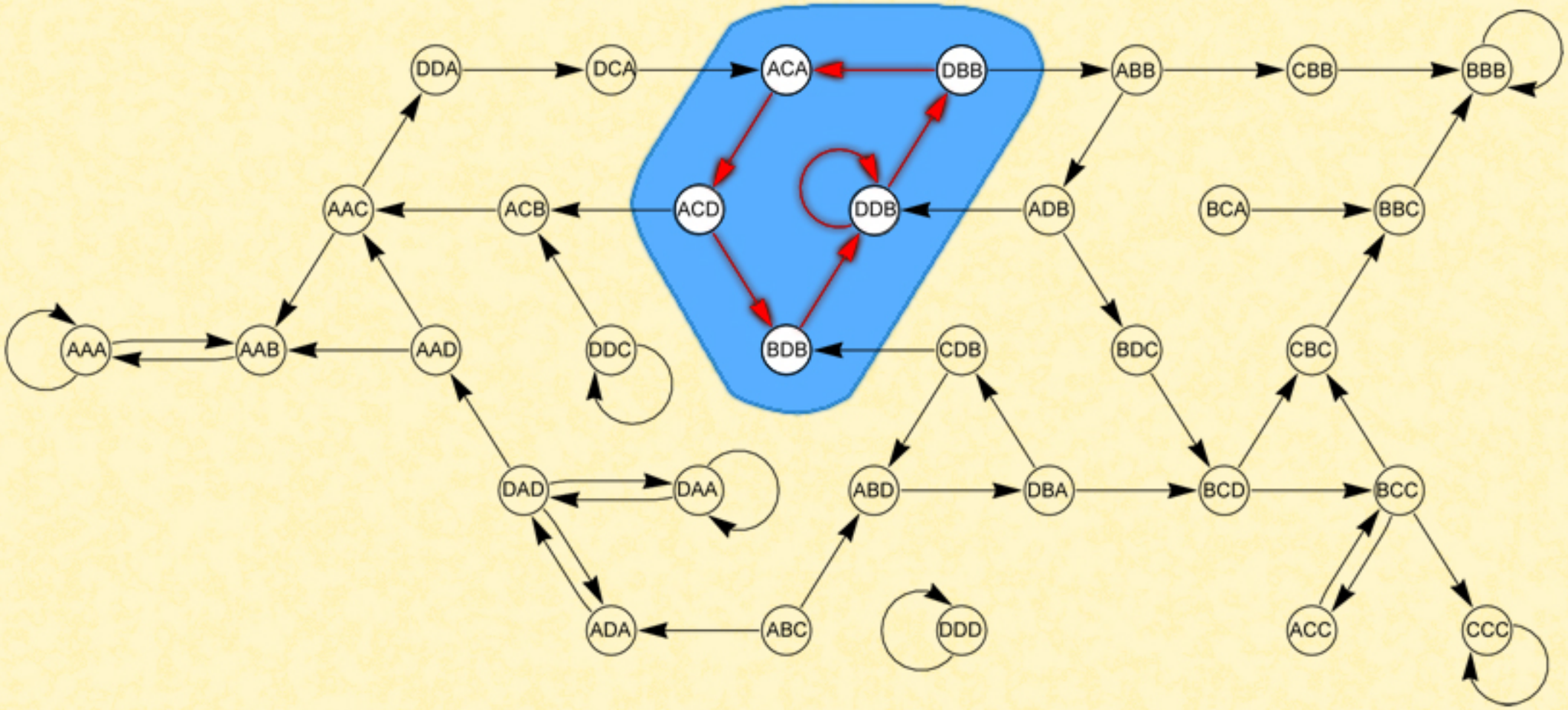
Resilient and Recoverable

- A state's resilience is the cumulative probability of returning to a state given that the system starts in that state: Sum over t $P(s_t = S_i | s_0 = S_i)$.
- **Set** resilience is the probability that the system will return to a set if the initial state of a sequence is within the state.
- A transition out of the set is recoverable to the degree that the system will return to the set after the transition. ***S*** is recoverable from $\sim(S_i, S_j)$ to the degree calculated by the Sum over t $P(s_t$ in $\mathbf{S} | \sim(S_i, S_j)$ and S_i in ***S*** and S_j not in ***S***).



Reliable

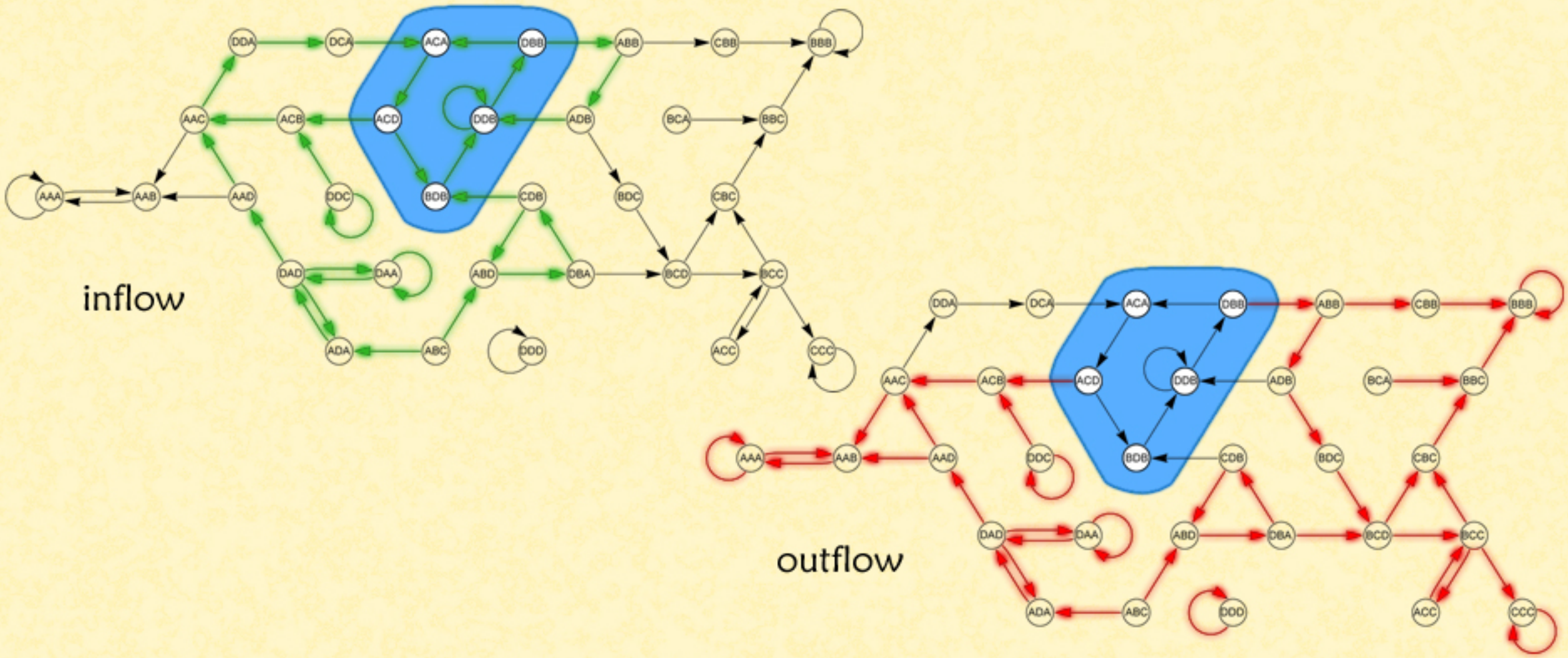
- The reliability of a set is the average cumulative long-term probability density over the states in the set given that the system starts within that set. It is by 1/|S| Sum over S_i, Sum over t P(s_t in **S**|s_0 in **S**).



Robust and Vulnerable

- The robustness of a set is the average cumulative long-term probability density over the states in the set given that the system may start at any state.
  $1/|S|$ Sum over S_i, Sum over t $P(s_t \text{ in } \boldsymbol{S})$.

- A set's vulnerability at S_i is the difference in the average long-term probability density over the states in the set compared to the density generated by starting in S_i: Sum over t $P(s_t \text{ in } \boldsymbol{S} \mid s_0 = S_i)$ − robustness of **S**.



Summary of Robustness Family

- Static - nothing changes
- Stable - dynamics don't lose the property
- Sustainability - the property is maintained next period
- Resilience - property is regained if lost
- Recoverable - property is regained it lost a certain way
- Reliability - the property is maintained
- Robustness - the property holds
- ...and don't forget turbulence.

Application and Expansion

- Develop packages for Java, Mathematica, Matlab, R, etc. to foster its usage by the research community.
- Through collaboration, apply to data and simulation output.
- Develop more and refined measures of the properties of system dynamics using this Markov representation.
- Use measure similarity to find equivalence classes for system dynamics that can categorize processes by behavioral features.
- Consider other representations for measures can't be captured this way; e.g. non-probabilistic measures of these phenomena.

Thank You

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